ACADEMIC PRESS

Available online at www.sciencedirect.com

SCIENCE DIRECT

Journal of Sound and Vibration 271 (2004) 47–66

JOURNAL OF SOUND AND VIBRATION

www.elsevier.com/locate/jsvi

Fundamental vibration of rotating cantilever blades with pre-twist

X.X. Hu[a,*], T. Sakiyama[b], H. Matsuda[b], C. Morita[b]

[a] *College of Mechanical Engineering, Zhejiang University of Technology, No. 6 District, Zhaohui Xincun, Hangzhou, Zhejiang 310032, China*

[b] *Department of Structural Engineering, Faculty of Engineering, Nagasaki University, 1-14 Bunkyomachi, Nagasaki 852-8521, Japan*

Received 28 May 2002; accepted 24 February 2003

Abstract

A non-linear strain–displacement relationship of a pre-twisted conical shell on the general shell theory is utilized, and a method for vibration of a rotating cantilever conical shell with pre-twist is developed by the principle of virtual work and the Rayleigh–Ritz method. Firstly, deformation and stress resultants caused by rotation are analyzed. Secondly, an equilibrium of energy for vibration of a pre-twisted conical shell having the conditions achieved in the first process is given and then an eigenfrequency equation of a rotating cantilever conical shell with pre-twisted is formulated. The effects of parameters such as an angular velocity, a radius of a hub, a setting angle, a twist angle, a subtended angle and a tapered ratio of cross-section on the fundamental vibration are investigated.

1. Introduction

A cantilever shell is a model of blades that appear in the aerospace, turbomachinery, and other engineering such as aerial propellers and turbofans. On the one hand, the complexity of configuration such as twist, curvature, non-uniform cross-section, and thickness makes the researches of blades difficult. But on the other it is quite significant for design, safety and life of machinery to determinate their dynamic characteristics accurately because they are working at high speed, which prompts researchers to devote their greater efforts on the problem.

From the previous research work of the blades, it is known that a beam was often used in study on the dynamics of blades in past decades [1–3] although it is the simplest model of blades, and it

*Corresponding author. Tel.: +86-571-88320472; fax: +86-571-88320130.

E-mail address: kokaka2sakura@hotmail.com (X.X. Hu).

0022-460X/03/$ - see front matter
doi:10.1016/S0022-460X(03)00262-1

is also found in the recent researches of the problems such as linear or non-linear vibration considering shear deformation, non-uniform cross-section, pre-twist, coupled vibration and concentrated mass [4–8]. Comparing with the beam model, a plate and a shell are more approximate models of blades, but there is a little work done. Ramamurti and Kielb [9] studied rotating twisted plates by FEM with two different shape functions. Sreenivasamurthy and Ramamurti [10] studied rotating pre-twisted and tapered plates by FEM with triangular shell elements having three nodes and 18 degrees of freedom respectively. Leissa et al. [11] studied cylindrical shells with camber and twist on the shallow shell theory by the Ritz method. Recently, Tsuiji et al. [12] studied rotating thin twisted plates by the Rayleigh–Ritz method, and Hu and Tsuiji [13] studied rotating cylindrical thin panels with twist on the general shell theory.

The main purpose in this paper is to extend our researches on twisted shells and to reveal vibration characteristics of the shells subjected axial forces such as centrifugal forces. Based on a non-linear relationship between strains and displacements for a pre-twisted conical shell derived by the Green strain tensor on the general shell theory, a numerical analysis of vibration is presented by the energy method. There are two processes in the numerical analysis, one is to analyze deformation and stress resultants in a pre-twisted conical shell induced by rotation, another is to formulate a governing equation for vibration of the rotating conical shell and then study the vibration characteristics under considering the results achieved in the first process as the initial conditions. The effects of parameters of this system such as an angular velocity, a radius of a hub, a setting angle, a twist angle, a subtended angle and a tapered ratio of cross-section on vibration characteristics of rotating cantilever conical shells with pre-twist are investigated by the present method briefly.

2. Theoretical analysis

As shown in Fig. 1 that a pre-twisted conical shell is fixed on a periphery of a rigid hub at a setting angle ϕ, where a rectangular right handed Cartesian co-ordinate system is (X_1, Y_1, Z_1) fixed in the space with O_0 at the center of the rotating hub and another Cartesian co-ordinate system is $(x, y, \tilde{z})$ with a set of unit vectors $(\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3)$. K represents a twist angle at a free end of the conical shell defined as $K = kl$ where the small k is a constant twist ratio. a, b and β are three parameters

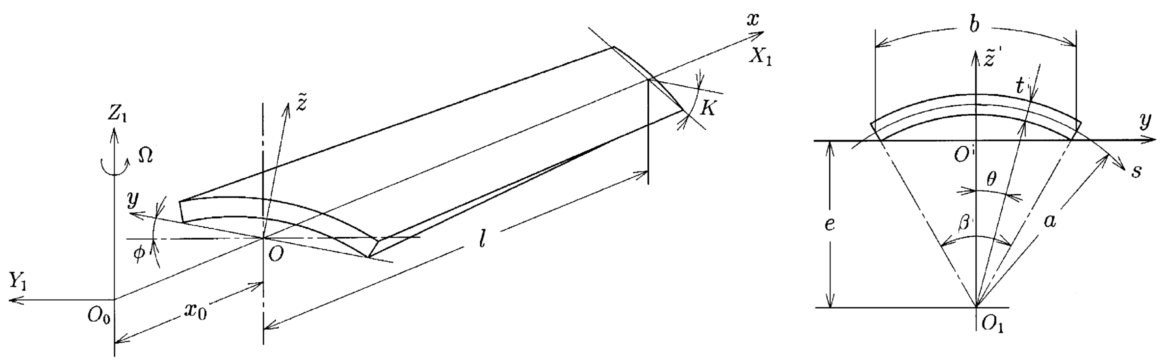

Fig. 1. A schematic diagram of a simple rotating blade system.

of a cross-section along the lengthwise direction, among them a subtended angle β is a constant, and a radius a and an arc-length b are variables. A tapered ratio of a conical shell is defined as $\alpha = b_1/b_0$ where b_0 and b_1 denote the parameters at a fixed and a free ends, respectively. The others are given in the figure.

2.1. Strain measures on shell theory

Let the mid-surface of the conical shell be taken as a reference surface which could be defined by two curvilinear coordinates $\alpha_1(\alpha_1 = x)$ and $\alpha_2(\alpha_2 = a\theta)$ with vectors $\mathbf{a}_1$ and $\mathbf{a}_2$, respectively, then a unit vector $\mathbf{a}_3$ perpendicular to the reference surface is chosen so that they form a local right-handed orthogonal co-ordinates, or

$$\mathbf{a}_1(\alpha_1, \alpha_2) = \frac{\partial \mathbf{r}_0^{(0)}}{\partial \alpha_1}, \quad \mathbf{a}_2(\alpha_1, \alpha_2) = \frac{\partial \mathbf{r}_0^{(0)}}{\partial \alpha_2}, \quad \mathbf{a}_3(\alpha_1, \alpha_2) = \frac{\mathbf{a}_1 \times \mathbf{a}_2}{|\mathbf{a}_1 \times \mathbf{a}_2|}, \tag{1}$$

where $\mathbf{r}_0^{(0)}$ denotes a position vector of a point in the reference surface.

To a point P outside the reference surface with a distance α_3, its position vector $\mathbf{r}^{(0)}$ can be given by

$$\mathbf{r}^{(0)} = \mathbf{r}_0^{(0)} + \alpha_3 \mathbf{a}_3. \tag{2}$$

The shell is assumed to be subjected to deformation and the displacement vector of the point P is $\mathbf{U}$, then the corresponding position vector $\mathbf{r}$ is

$$\begin{aligned} \mathbf{r} &= \mathbf{r}^{(0)} + \mathbf{U}(\alpha_1, \alpha_2, \alpha_3) \\ &= \mathbf{r}^{(0)} + U_1\mathbf{a}_1 + V_1\mathbf{a}_2 + W_3\mathbf{a}_3, \end{aligned} \tag{3}$$

where U_1, V_1 and W_1 denote the displacement components of $\mathbf{U}$ in the directions $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$, respectively. The covariant base vectors before deformation are denoted by $\mathbf{g}_i$ $(i = 1, 2, 3)$, and those after deformation by $\mathbf{G}_i$ $(i = 1, 2, 3)$, namely,

$$\mathbf{g}_i = \frac{\partial \mathbf{r}^{(0)}}{\partial \alpha_i}, \quad \mathbf{G}_i = \frac{\partial \mathbf{r}}{\partial \alpha_i} \quad (i = 1, 2, 3). \tag{4}$$

A strain measure with respect to the general curvilinear co-ordinates $(\alpha_1, \alpha_2, \alpha_3)$ can be given by the Green strain tensor as follows:

$$2f_{ij} = \mathbf{G}_i \cdot \mathbf{G}_j - \mathbf{g}_i \cdot \mathbf{g}_j \quad (i,j = 1, 2, 3). \tag{5}$$

Considering the Kirchhoff–Love Hypothesis for the pre-twisted thin conical shell, the following equations can be achieved:

$$\bar{U} = u - \frac{z}{g}\left(\frac{p_{,x}}{\sqrt{g}}u + \frac{p_{,\theta}}{a\sqrt{g}}v + \frac{\partial w}{\partial x} - \frac{q}{a}\frac{\partial w}{\partial \theta}\right),$$

$$\bar{V} = v - \frac{z}{\sqrt{g}}\left[\left(k - \frac{p_{,x}q}{g}\right)u - \frac{1}{a}\left(1 + \frac{p_{,\theta}q}{g}\right)v - \frac{q}{\sqrt{g}}\frac{\partial w}{\partial x} + \frac{\sqrt{g}}{a}\left(1 + \frac{q^2}{g}\right)\frac{\partial w}{\partial \theta}\right],$$

$$\bar{W} = w,$$

$$\mathbf{U} = \left(\bar{U} - \frac{p}{\sqrt{g}}\bar{W}\right)\mathbf{i}_1 + \left(f\bar{U} + \bar{V}\cos\theta + \frac{\sin\theta}{\sqrt{g}}\bar{W}\right)\mathbf{i}_2 + \left(h\bar{U} - \bar{V}\sin\theta + \frac{\cos\theta}{\sqrt{g}}\bar{W}\right)\mathbf{i}_3, \tag{6}$$

which means that the displacements $\bar{U}$, $\bar{V}$ and $\bar{W}$ of an arbitrary point can be defined by the displacements u, v and w of the corresponding point on the reference surface. The notations $(\,)_{,x}$ and $(\,)_{,\theta}$ express partial differentiation with respect to x and θ, respectively. The variables in the paper are given in Appendix A.

Thereby, the engineering strains with respect to a local orthogonal co-ordinate system (ξ, η, ζ) at the point P are obtained by

$$\begin{Bmatrix} \varepsilon_{\xi\xi} \\ \varepsilon_{\eta\eta} \\ \gamma_{\xi\eta} \end{Bmatrix} = \frac{1}{1 - \bar{m}\dfrac{z}{l}}\mathbf{Z}\mathbf{G}\bar{\mathbf{U}} + \frac{1}{2}\begin{Bmatrix} \bar{\mathbf{U}}^{\mathrm{T}}\mathbf{G}_x^{\mathrm{T}}\mathbf{G}_x\bar{\mathbf{U}} \\ \bar{\mathbf{U}}^{\mathrm{T}}\mathbf{G}_\theta^{\mathrm{T}}\mathbf{G}_\theta\bar{\mathbf{U}} \\ \bar{\mathbf{U}}^{\mathrm{T}}\mathbf{G}_x^{\mathrm{T}}\mathbf{G}_\theta\bar{\mathbf{U}} \end{Bmatrix}, \tag{7}$$

where

$$\mathbf{Z} = \begin{bmatrix} 1 & \frac{z}{l} & \frac{z^2}{l^2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{z}{l} & \frac{z^2}{l^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{z}{l} & \frac{z^2}{l^2} \end{bmatrix}, \quad \mathbf{G} = [G_{i,j}], \ (i = 1 \sim 9;\ j = 1 \sim 12),$$

$$\bar{\mathbf{U}}^T = \left[\frac{\partial U}{\partial X} \quad \frac{\partial U}{\partial \theta} \quad U \quad \frac{\partial V}{\partial X} \quad \frac{\partial V}{\partial \theta} \quad V \quad \frac{\partial^2 W}{\partial X^2} \quad \frac{\partial^2 W}{\partial \theta^2} \quad \frac{\partial^2 W}{\partial X \partial \theta} \quad \frac{\partial W}{\partial X} \quad \frac{\partial W}{\partial \theta} \quad W\right],$$

$$\mathbf{G}_x = \left[0 \quad 0 \quad \frac{\bar{p}_{,x}}{g} \quad 0 \quad 0 \quad \frac{p_{,\theta}}{\bar{a}g} \quad 0 \quad 0 \quad 0 \quad \frac{1}{\sqrt{g}} \quad -\frac{q}{\bar{a}\sqrt{g}} \quad 0\right],$$

$$\mathbf{G}_\theta = \left[0 \quad 0 \quad \frac{K}{\sqrt{g}} \quad 0 \quad 0 \quad -\frac{1}{\bar{a}\sqrt{g}} \quad 0 \quad 0 \quad 0 \quad 0 \quad \frac{1}{\bar{a}} \quad 0\right], \tag{8}$$

and the non-zero elements in matrix $\mathbf{G}$ are defined in Appendix B.

The variables in Eqs. (7) and (8) are dimensionless ones because the following dimensionless indices are introduced:

$$U = \frac{u}{l}, \quad V = \frac{v}{l}, \quad W = \frac{w}{l}, \quad X = \frac{x}{l},$$

$$K = kl, \quad \bar{a} = \frac{a}{l}, \quad \bar{e} = \frac{e}{l}, \quad \bar{d}_2 = d_2 l, \quad \bar{d}_3 = d_3,$$

$$\bar{m} = ml, \quad \bar{p}_{,x} = p_{,x}l, \quad \bar{q}_{,x} = q_{,x}l, \tag{9}$$

2.2. Deformation and stress resultants by rotation

Thin conical shells are studied in here, therefore, it can be assumed that a centrifugal force is a constant throughout the thickness and the position vector $\mathbf{r}$ can be rewritten as

$$\mathbf{r} = \left(\begin{Bmatrix} x_0 + x \\ a\sin\theta \\ a\cos\theta - e \end{Bmatrix}^{\mathrm{T}} + \begin{Bmatrix} u - \dfrac{p}{g}w \\ fu + v\cos\theta + \dfrac{\sin\theta}{\sqrt{g}}w \\ hu - v\sin\theta + \dfrac{\cos\theta}{\sqrt{g}}w \end{Bmatrix}^{\mathrm{T}} \right) \begin{Bmatrix} \mathbf{i}_1 \\ \mathbf{i}_2 \\ \mathbf{i}_3 \end{Bmatrix}. \tag{10}$$

Neglected Coriolis effects, the D'Alembert force $\mathbf{F}$ per unit volume in the conical shell which rotates at an angular velocity Ω is given by

$$\begin{aligned}
\mathbf{F} = & -\rho\frac{\partial^2\mathbf{r}}{\partial t_{ime}^2} = \mathbf{F}_e + \mathbf{F}_c \\
= & -\omega^2\rho \begin{Bmatrix} u - \dfrac{p}{g}w \\ fu + v\cos\theta + \dfrac{\sin\theta}{\sqrt{g}}w \\ hu - v\sin\theta + \dfrac{\cos\theta}{\sqrt{g}}w \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \mathbf{i}_1 \\ \mathbf{i}_2 \\ \mathbf{i}_3 \end{Bmatrix} \\
& - \Omega^2\rho \left(\left(\begin{Bmatrix} x_0 + x \\ a\cos\theta\sin(\theta-\psi) + e\sin\theta\cos\theta \\ -a\sin\theta\sin(\theta-\psi) - e\sin^2\theta \end{Bmatrix}^{\mathrm{T}} + \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}^{\mathrm{T}} \right. \right. \\
& \left. \left. \begin{bmatrix} 1 & (f\cos\theta - h\sin\psi)\cos\psi & (h\sin\psi - f\cos\theta)\sin\psi \\ 0 & \cos\psi\cos(\theta-\psi) & -\sin\psi\cos(\theta-\psi) \\ -\dfrac{p}{\sqrt{g}} & \dfrac{\cos\psi}{\sqrt{g}}\sin(\theta-\psi) & -\dfrac{\sin\psi}{\sqrt{g}}\sin(\theta-\psi) \end{bmatrix} \right) \right) \begin{Bmatrix} \mathbf{i}_1 \\ \mathbf{i}_2 \\ \mathbf{i}_3 \end{Bmatrix},
\end{aligned} \tag{11}$$

where $\mathbf{F}_e$ and $\mathbf{F}_c$ denote an inertia force of vibration and a centrifugal force, respectively. ω is a natural frequency of vibration, ρ a density of a material, and $\psi = \phi + kx$.

As the centrifugal force is considered alone, the following equation can be given by the principle of virtual work:

$$0 = \int\int\int_{vol} \left(\delta[\, \varepsilon^L_{\xi\xi} \quad \varepsilon^L_{\eta\eta} \quad \gamma^L_{\xi\eta} \,] \begin{Bmatrix} \sigma_{\xi\xi} \\ \sigma_{\eta\eta} \\ \tau_{\xi\eta} \end{Bmatrix} + \delta[\, \varepsilon^N_{\xi\xi} \quad \varepsilon^N_{\eta\eta} \quad \gamma^N_{\xi\eta} \,] \begin{Bmatrix} \sigma^{(0)}_{\xi\xi} \\ \sigma^{(0)}_{\eta\eta} \\ \tau^{(0)}_{\xi\eta} \end{Bmatrix} \right)$$
$$\times \sqrt{g}\left(1 - \bar{m}\frac{z}{l}\right) \mathrm{d}xa\, \mathrm{d}\theta\, \mathrm{d}z$$
$$+ \int\int\int_{vol} \mathbf{F}_c \delta \mathbf{U} \sqrt{g}\left(1 - \bar{m}\frac{z}{l}\right) \mathrm{d}xa\, \mathrm{d}\theta\, \mathrm{d}z, \tag{12}$$

where $\sigma^{(0)}_{\xi\xi}$, $\sigma^{(0)}_{\eta\eta}$ and $\tau^{(0)}_{\xi\eta}$ denote the stresses produced by the centrifugal force.

Substituting the related quantities into Eq. (12), integrating with respect to z, multiplying by l^2/D, and then readjusting the equation yield

$$\int\int_S \delta\bar{\mathbf{U}}^{\mathrm{T}}[\mathbf{G}^{\mathrm{T}}\bar{D}\mathbf{G} + \bar{N}^{(0)}_{\xi\xi}\mathbf{G}^{\mathrm{T}}_x\mathbf{G}_x + \bar{N}^{(0)}_{\eta\eta}\mathbf{G}^{\mathrm{T}}_\theta\mathbf{G}_\theta + \bar{N}^{(0)}_{\xi\eta}(\mathbf{G}^{\mathrm{T}}_x\mathbf{G}_\theta + \mathbf{G}^{\mathrm{T}}_\theta\mathbf{G}_x)]\bar{\mathbf{U}}\sqrt{g}\bar{a}\, \mathrm{d}X\, \mathrm{d}\theta$$
$$- \bar{\Omega}^2\lambda_0^2 \int\int_S [(\alpha_{11}U + \alpha_{12}V + \alpha_{13}W)\delta U + (\alpha_{21}U + \alpha_{22}V + \alpha_{23}W)\delta V + (\alpha_{31}U + \alpha_{32}V$$
$$+ \alpha_{33}W)\delta W]\sqrt{g}\bar{a}\, \mathrm{d}X\, \mathrm{d}\theta$$
$$= \bar{\Omega}^2\lambda_0^2 \int\int_S (\bar{\beta}_1\delta U + \bar{\beta}_2\delta V + \bar{\beta}_3\delta W)\sqrt{g}\bar{a}\, \mathrm{d}X\, \mathrm{d}\theta, \tag{13}$$

where

$$\bar{\mathbf{D}} = \begin{bmatrix} \bar{\mathbf{D}}_1 & v\bar{\mathbf{D}}_1 & \mathbf{0} \\ & \bar{\mathbf{D}}_1 & \mathbf{0} \\ Sym. & & \dfrac{1-v}{2}\bar{\mathbf{D}}_1 \end{bmatrix}, \quad \bar{\mathbf{D}}_1 = \begin{bmatrix} \bar{H} & \bar{m} & 1 \\ & 1 & 0 \\ Sym. & & 0 \end{bmatrix}, \quad \bar{H} = \frac{Et}{1-v^2}\frac{l^2}{D}, \quad D = \frac{Et^3}{12(1-v^2)},$$

$$\bar{\Omega} = \frac{\Omega}{\omega_0}, \quad \left[\, \bar{N}^{(0)}_{\xi\xi} \quad \bar{N}^{(0)}_{\eta\eta} \quad \bar{N}^{(0)}_{\xi\eta} \,\right] = \frac{l^2}{D}\int_{-t/2}^{t/2}\left[\, \sigma^{(0)}_{\xi\xi} \quad \sigma^{(0)}_{\eta\eta} \quad \tau^{(0)}_{\xi\eta} \,\right]\left(1 - \bar{m}\frac{z}{l}\right) \mathrm{d}z. \tag{14}$$

ω_0 is a reference frequency and λ_0 is a reference frequency parameter defined as the same as the λ. α_{ij} $(i,j = 1,2,3)$ and $\bar{\beta}_i$ $(i = 1,2,3)$ are defined in Appendix C.

According to the demand of the Rayleigh–Ritz method, the displacement components U, V and W should satisfy the geometric boundary conditions which are assumed to be as two-dimensional algebraic polynomial functions,

$$U = \sum_{i=1}^{N_u}\sum_{j=0}^{M_u} a'_{ij}X^i\theta^j, \quad V = \sum_{k=1}^{N_v}\sum_{l=0}^{M_v} b'_{kl}X^k\theta^l, \quad W = \sum_{m=2}^{N_w}\sum_{n=0}^{M_w} c'_{mn}X^m\theta^n, \tag{15}$$

where a'_{ij}, b'_{kl} and c'_{mn} are unknown coefficients.

Substituting Eq. (15) into Eq. (13) and integrating over the surface area of the conical shell, the following algebraic equations are achieved:

$$\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{A}_{13} \\ & \mathbf{A}_{22} & \mathbf{A}_{23} \\ Sym. & & \mathbf{A}_{33} \end{bmatrix} \begin{Bmatrix} \mathbf{a}' \\ \mathbf{b}' \\ \mathbf{c}' \end{Bmatrix} = \begin{Bmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \end{Bmatrix}, \tag{16}$$

where $\mathbf{A}_{ij}$ $(i,j = 1,2,3)$ are stiffness matrices included inherent, geometric and supplementary stiffness, in which the geometric stiffness depends on the deformation and stress resultants by rotation. $\mathbf{P}_i$ $(i = 1,2,3)$ are force vectors.

It is seen that the unknown coefficients a'_{ij}, b'_{kl} and c'_{mn} can be solved from Eq. (16), then the deformation and the stress resultants due to rotation can be obtained in turn.

2.3. Free vibration of rotating conical shells

Let us assume the deformation and the stress resultants as initial conditions and then analyze vibration by the same method as aforementioned. The principle of virtual work for free vibration of a rotating pre-twisted conical shell is given as the following non-dimensional equation:

$$\begin{aligned} &\int\int_S \delta\mathbf{U}^{\mathrm{T}}[\mathbf{G}^{\mathrm{T}}\bar{D}\mathbf{G} + \bar{N}^{(0)}_{\xi\xi}\mathbf{G}_x^{\mathrm{T}}\mathbf{G}_x + \bar{N}^{(0)}_{\eta\eta}\mathbf{G}_\theta^{\mathrm{T}}\mathbf{G}_\theta + \bar{N}^{(0)}_{\xi\eta}(\mathbf{G}_x^{\mathrm{T}}\mathbf{G}_\theta + \mathbf{G}_\theta^{\mathrm{T}}\mathbf{G}_x)]\mathbf{U}\sqrt{g}\bar{a}\,\mathrm{d}X\,\mathrm{d}\theta \\ &\quad - \bar{\Omega}^2\lambda_0^2\int\int_S[(\alpha_{11}U + \alpha_{12}V + \alpha_{13}W)\delta U + (\alpha_{21}U + \alpha_{22}V + \alpha_{23}W)\delta V \\ &\quad + (\alpha_{31}U + \alpha_{32}V + \alpha_{33}W)\delta W]\sqrt{g}\bar{a}\,\mathrm{d}X\,\mathrm{d}\theta \\ &\quad\quad = -\lambda^2\int\int_S[(g + q^2)U\delta U + qU\delta V + qV\delta U + V\delta V + W\delta W]\sqrt{g}\bar{a}\,\mathrm{d}X\,\mathrm{d}\theta, \end{aligned} \tag{17}$$

where λ is a vibration frequency parameter defined by

$$\lambda^2 = \frac{12(1 - v^2)\rho\omega^2 l^4}{Et^2}. \tag{18}$$

Further, let Eq. (15) into Eq. (17), an eigenfrequency equation is given in a matrix form, or

$$\begin{bmatrix} \mathbf{A}_{11} - \lambda^2\mathbf{B}_{11} & \mathbf{A}_{12} - \lambda^2\mathbf{B}_{12} & \mathbf{A}_{13} \\ & \mathbf{A}_{22} - \lambda^2\mathbf{B}_{22} & \mathbf{A}_{23} \\ Sym. & & \mathbf{A}_{33} - \lambda^2\mathbf{B}_{33} \end{bmatrix} \begin{Bmatrix} \mathbf{a}' \\ \mathbf{b}' \\ \mathbf{c}' \end{Bmatrix} = \mathbf{0}, \tag{19}$$

where $\mathbf{B}_{ij}$ $(i,j = 1,2,3)$ denote mass matrices.

3. Numerical results and analyses

As shown in previous section, it is an iterative procedure for solving the deformation and strain resultants caused by rotation, a criterion is needed for the procedure control. In here, the frequency parameter λ is considered as a controlled quantity which satisfies the following equation:

$$\frac{\left|\lambda_i^{(j+1)} - \lambda_i^{(j)}\right|}{\lambda_i^{(j+1)}} \leqq 0.1\% \quad (i = 1 \sim 8), \tag{20}$$

where i denotes the ith vibration mode and j denotes the jth iteration.

The numerical integration method in this paper is Gauss–Legendre method with 16 integration points that can ensure the solutions of this kind problem convergence. The Poisson ratio v is 0.3.

3.1. Convergence of frequency parameters

From Eq. (15), it is known that a proper choice of the numbers of terms in the displacement functions is significant for approximate solutions. For simplicity, it is assumed that the maximum powers of two variables in U and V functions are the same, because they are in-plane displacement components, or $N_u = N_v$ and $M_u = M_v$. A blade, whose parameters are a thickness ratio $b_0/t = 25$, an aspect ratio $l/b_0 = 2.0$, a tapered ratio $\alpha = 0.6$, a subtended angle $\beta = 60°$, a twist angle $K = 60°$, a setting angle $\phi = 45°$, a radius of a hub $X_0 = 2.0$ and an angular velocity $\bar{\Omega} = 1.0$, is analyzed by the present method and the results for the combinations of various terms are shown in Table 1. A good rate of convergence is obtained as the maximum powers of X and θ are 8 and 7 in U and V functions, respectively, and 8 and 8 in W function. Herein, the above choice for the displacement functions is adopted in the following analyses.

3.2. Accuracy of the present method

As we know that there were no researches on rotating twisted and open conical shells, so a rotating twisted cylindrical thin panels, which is a typical conical shell with a tapered ratio of cross-section ($\alpha = 1.0$), is considered for demonstrating the accuracy and practicability of the present method. The parameters are given as the follows: $b_0/t = 20$, $b_0/l = 0.5$, $K = 0°$ and $60°$, $\beta = 30°$ and $60°$, $\phi = 0°$, $X_0 = 0.0$ and $\bar{\Omega} = 0.8$. The first eight frequency parameters obtained by

Table 1
Convergence of λ_i for various number of terms in displacement functions ($b_0/t = 25, l/b_0 = 2.0, \alpha = 0.6, \beta = 60°, K = 60°, \phi = 45°, X_0 = 2.0, \bar{\Omega} = 1.0$)

N_u, M_u	7, 6	7, 7	7, 7	7, 7	8, 7	8, 7
N_v, M_v	7, 6	7, 7	7, 7	7, 7	8, 7	8, 7
N_w, M_w	8, 7	8, 7	8, 8	9, 8	8, 8	9, 8
Terms	49/49/56	56/56/56	56/56/63	56/56/72	64/64/63	64/64/72
1	9.4108	9.4936	9.4105	9.4090	9.4096	9.4083
2	27.947	28.379	27.946	27.941	27.940	27.936
3	47.252	47.538	47.251	47.243	47.242	47.236
4	71.986	73.067	71.982	71.978	71.963	71.950
5	101.41	102.98	101.41	101.40	101.32	101.30
6	117.18	117.73	117.17	117.15	116.95	116.91
7	136.51	136.96	136.48	136.44	135.81	135.72
8	157.74	157.92	157.69	157.64	157.04	156.90

Table 2
Comparisons of frequency parameters λ_i for rotating cylindrical thin panels ($b_0/t = 20, l/b_0 = 2.0, \alpha = 1.0, \phi = 0°, X_0 = 0.0, \bar{\Omega} = 0.8$)

K	0°				60°			
β	30°		60°		30°		60°	
Method	Ref. [13]	Present	Ref. [13]	Present	Ref. [13]	Present	Ref. [13]	Present
1	6.2984	6.2977	9.6696	9.6677	4.8728	4.8865	6.2134	6.2030
2	15.393	15.393	15.560	15.562	17.230	17.258	19.393	19.391
3	32.141	32.139	49.260	49.254	31.351	31.246	36.144	35.951
4	49.324	49.324	50.770	50.770	52.177	52.178	54.311	54.339
5	57.471	57.457	57.628	57.614	75.390	75.175	76.140	75.843
6	78.307	78.303	90.201	90.200	87.821	87.787	91.338	91.266
7	92.972	92.972	100.06	100.05	94.268	94.135	105.91	105.35
8	95.100	95.098	108.40	108.39	109.40	108.99	109.19	109.02

the present method and provided by a Ref. [13] are shown in Table 2. It is observed that they have a good agreement and the maximum difference between them is less than 1%.

3.3. Effects on fundamental vibration

In this section, a blade is considered as a cantilever pre-twisted conical shell. The general parameters are given as follows: $K = 30°$, $\beta = 60°$, $\phi = 0°$, $X_0 = 0.5$, $l/b_0 = 2.0$, $b_0/t = 20$ and 40, and $\alpha = 0.6$, which means when a effect of a parameter is investigated the others remain the above values. The angular velocity $\bar{\Omega}$ always changes from 0.0 to 3.0.

An effect of a twist angle on the fundamental frequency parameter λ_1 is studied and the results are given in Fig. 2. In general, the fundamental frequency parameter decreases with a twist angle increasing and increases with an increase in an angular velocity $\bar{\Omega}$ for the conical shells with two thickness ratios $b_0/t = 20$ and 40. It is obviously that the thin conical shell is more sensitive to the twist angle than the thick one, or the fundamental frequency parameter decreases greatly with an increase in a twist angle for the thin conical shell. For instance at $\bar{\Omega} = 0.0$, the variations of λ_1 are 22.16% for the thin conical shell and only 4.88% for the thick one with the twist angle rising from 0° to 15°, and those are 97.93% and 31.89% when the K changes from 0° to 45°. But with the change in the angular velocity the increase of the fundamental frequency parameter of the thick conical shell is greater than that of the thin one for various twist angles. In the case of $\bar{\Omega}$ rising from 0.0 to 3.0 and the various twist angles $K = 0°$, 15°, 30° and 45°, the increases in λ_1 of the conical shell with $b_0/t = 20$ are 71.45%, 78.04%, 92.65% and 107.08%, respectively, and 24.05%, 42.33%, 63.34% and 79.25% for the other with $b_0/t = 40$. Otherwise, it is seen that the differences caused by various twist angle decrease with an increase in the angular velocity. From the above, it is known that a centrifugal force of rotation may cause deformation against the pre-twisted deformation that is large for a thick and highly pre-twisted conical shell. There is an obvious phenomenon in Table 3 that the number of iteration for solving convergent frequency parameters increases with the twist angle and the angular velocity increasing, which could be explained by the anti-deformation due to the pre-twist and the centrifugal force.

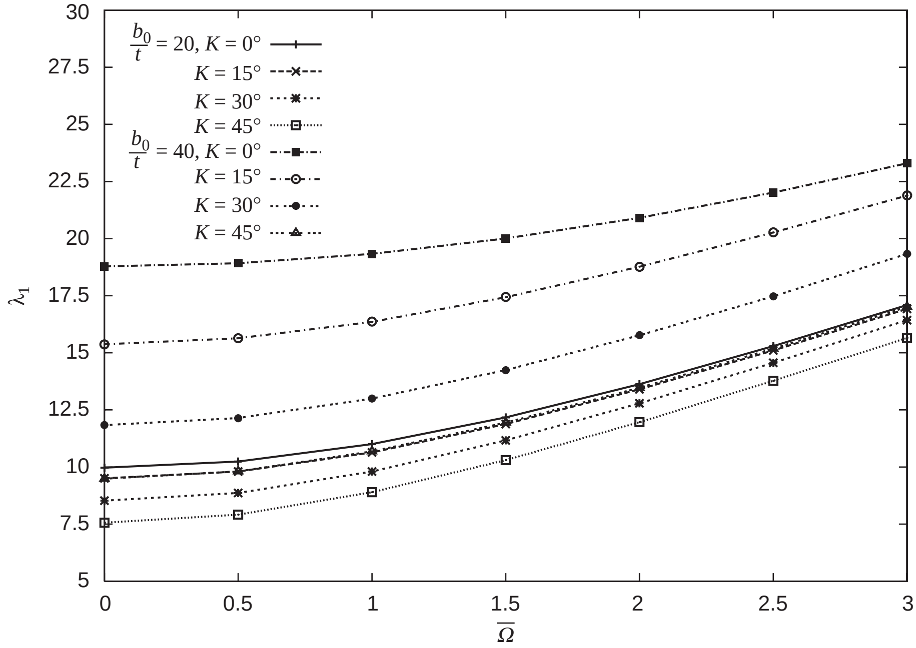

Fig. 2. Variation of λ_1 versus $\bar{\Omega}$ on various K and b_0/t.

Table 3
Effect of K on iterations of calculation ($b_0/t = 20, l/b_0 = 2.0, \alpha = 0.6, \beta = 60°, X_0 = 0.5, \phi = 0°$)

Ω	0.0	0.5	1.0	1.5	2.0	2.5	3.0
K	Iterations of calculation						
0°	3	3	3	3	4	5	7
15°	3	3	4	6	8	11	14
30°	3	3	5	7	10	14	18
45°	3	3	6	9	14	21	30

Fig. 3 depicts an influence of a setting angle ϕ for rotating conical shells without and with twist. A decrease tendency is found for the fundamental frequency parameter with an increase in a setting angle and it becomes greatly as an angular velocity increases in the combinations of K and b_0/t. The variations in the case of $K = 30°$ are larger than those in the case of $K = 0°$. For an example of $b_0/t = 20$ and $\phi = 0°$, the variations are 92.65% for $K = 30°$ and 71.45% for $K = 0°$ as $\bar{\Omega}$ rises from 0.0 to 3.0, but for a given twist angle the variations decrease with an increase in the setting angle, they are 92.65%, 79.71% and 58.15% in the three cases of $\phi = 0°$, 30° and 60°, respectively. To the conical shell with $b_0/t = 40$, the variation of the fundamental frequency parameter with the angular velocity $\bar{\Omega}$ is smaller than that of the other with $b_0/t = 20$ for the combinations of K and ϕ. The effect of the twist angle remains the same as aforementioned, or the fundamental frequency parameter decreases with an increase in the twist angle.

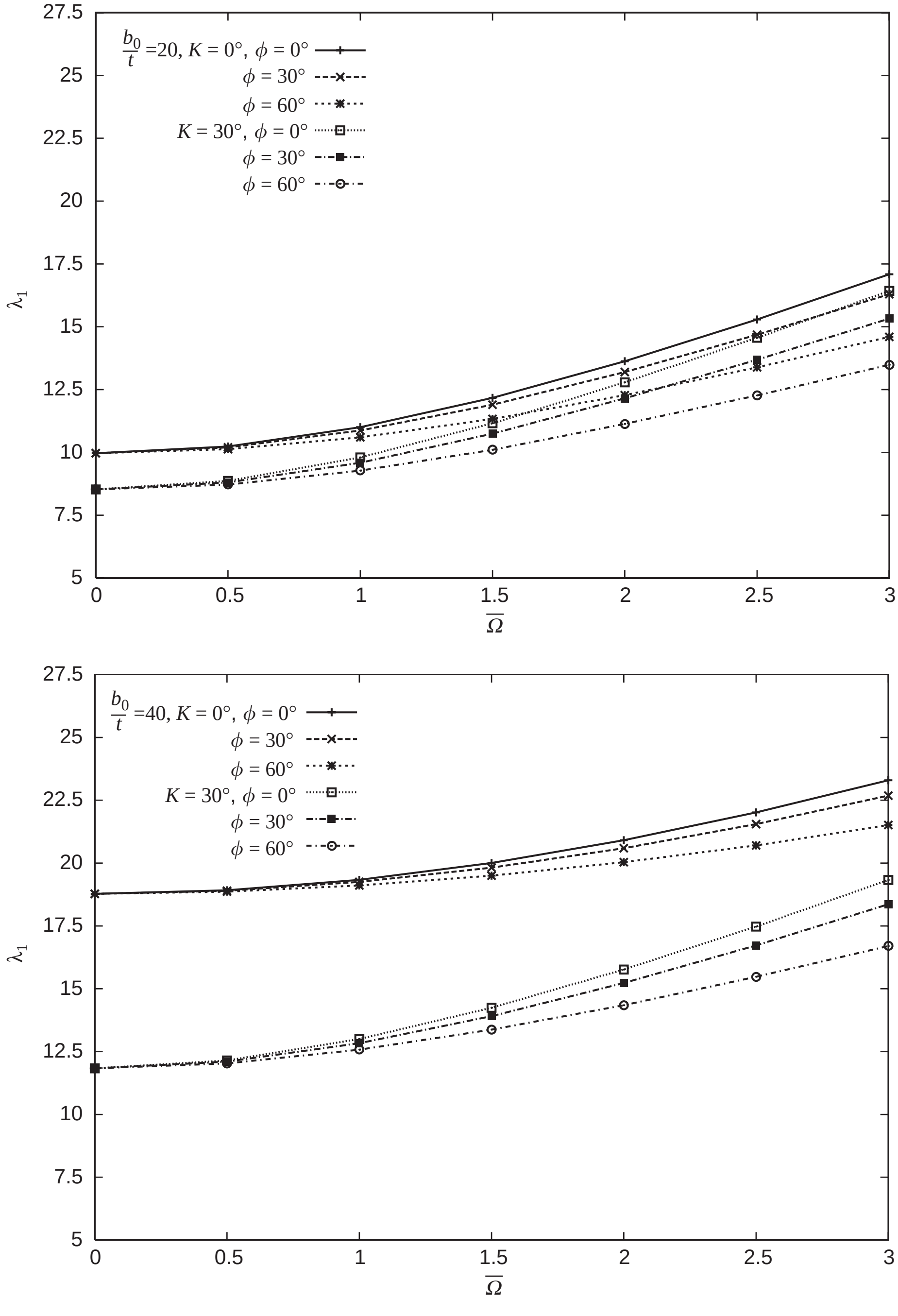

Fig. 3. Variation of λ_1 versus $\bar{\Omega}$ on various ϕ, K and b_0/t.

An effect of a subtended angle β on the fundamental frequency parameter is investigated for rotating conical shell with the same combinations of the thickness ratios and the twist angles as the above. It is known that an increase in the subtended angle means an increase in the curvature

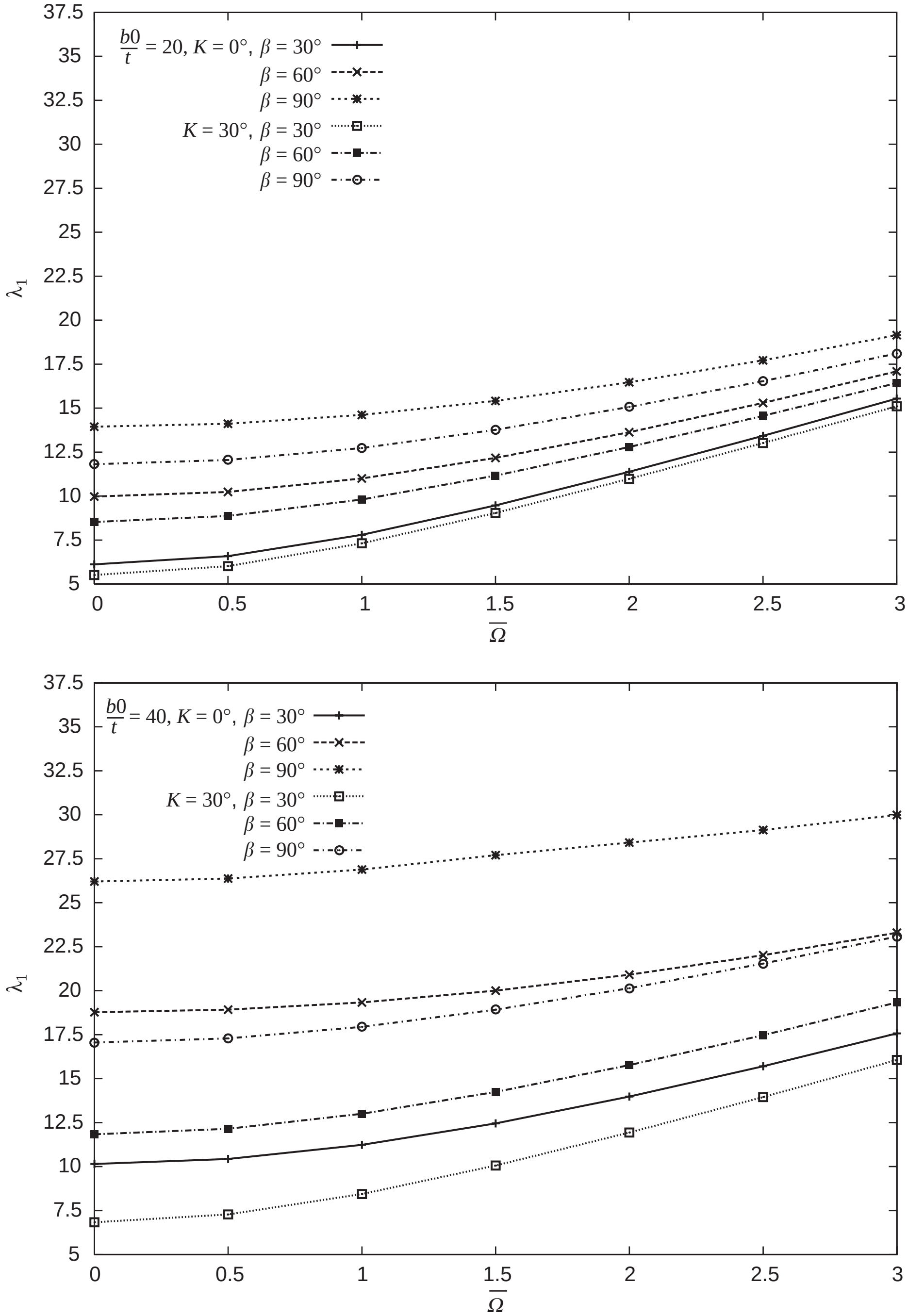

Fig. 4. Variation of λ_1 versus $\bar{\Omega}$ on various β, K and b_0/t.

in chordwise direction under a constant length of arc on an arbitrary cross-section, and then the stiffness of the conical shell raises, which is why the fundamental frequency parameter increases with β increasing as shown in Fig. 4. The influence of the subtended angle and the twist angle on the thin conical shell is greater than that on the thick one but the variation caused by rotation is contrary, where it increases with an increase in K and decreases with an increase in β. In the case of $K = 30^\circ$ and β rising from 30° to 60° and 90°, the variations of λ_1 are 73.21% and 149.62% for the thin shells, and 54.65% and 114.18% for the thick ones at $\bar{\Omega} = 0.0$. With $\bar{\Omega}$ varying from 0.0 to 3.0 the variations of λ_1 are 35.31% and 63.34% corresponding to $\beta = 60^\circ$ and 90° for thin conical shells with $K = 30^\circ$, and 53.28% and 92.65% for the thick ones. The effect decreases with $\bar{\Omega}$ increasing. In the case of $b_0/t = 20$ and $K = 0^\circ$, the variation reaches 127.81% at $\bar{\Omega} = 0.0$ and only 23.21% at $\bar{\Omega} = 3.0$ when the subtended angle β changes from 30° to 90°, respectively, the same trend is observed for the case of $K = 30^\circ$.

An influence of a tapered ratio of cross-section α on the fundamental frequency parameter is given in Fig. 5. It is observed that the fundamental frequency parameter decreases with an increase in α for combinations of b_0/t and K, because an increase of the tapered ratio makes the curvature decrease and then stiffness decrease under a condition of a constant subtended angle. The variation of the fundamental frequency parameter shows an increase with the angular velocity for a given α which is greater in the case of $K = 30^\circ$ than that in the case of $K = 0^\circ$. The effect of α on the thin conical shell is larger than on the thick one, which is the same as the influence of the twist angle. The curves of λ_1 with the angular velocity $\bar{\Omega}$ for a given b_0/t and K are almost parallel to each other, which means that the influence of α on λ_1 is not or hardly affected by the angular velocity.

A radius X_0 of a rotating hub, which is considered as a significant factor for the magnitude of a centrifugal force, is also investigated and the results are shown in Fig. 6. It is known that the centrifugal force increases with the radius X_0 increasing which leads large deformation and stress resultants in the conical shell, therefore the fundamental frequency parameter appears an increase that becomes large with an increase in the angular velocity. The presence of the twist angle makes λ_1 decrease, which is great for thin conical shell. But the increase with the $\bar{\Omega}$ is great for thick conical shells with both cases of K. For example of $\bar{\Omega}$ changing from 0.0 to 3.0 and $X_0 = 4.0$, the variations are 188.06% for $K = 0^\circ$ and 227.58% for $K = 30^\circ$ in the case of $b_0/t = 20$, they are 75.14% and 159.48% in the case of $b_0/t = 40$.

4. Conclusions

A numerical method for free vibration of rotating pre-twisted conical shells is proposed, where the strain measures including non-linear parts are derived by the Green strain tensor on the thin shell theory, and the equations are formulated by the principle of virtual work and the Rayleigh–Ritz method. There are two processes in the present method. Firstly, deformation and stress resultants are analyzed considering the centrifugal force induced by rotation. Secondly, the free vibration characteristics of the conical shell subjected the deformation and stress resultants obtained in the first process is achieved. The effects of various parameters such as a twist angle, a subtended angle, a setting angle, a tapered ratio, a radius of a hub and a angular velocity of rotation on the fundamental frequency parameter are investigated.

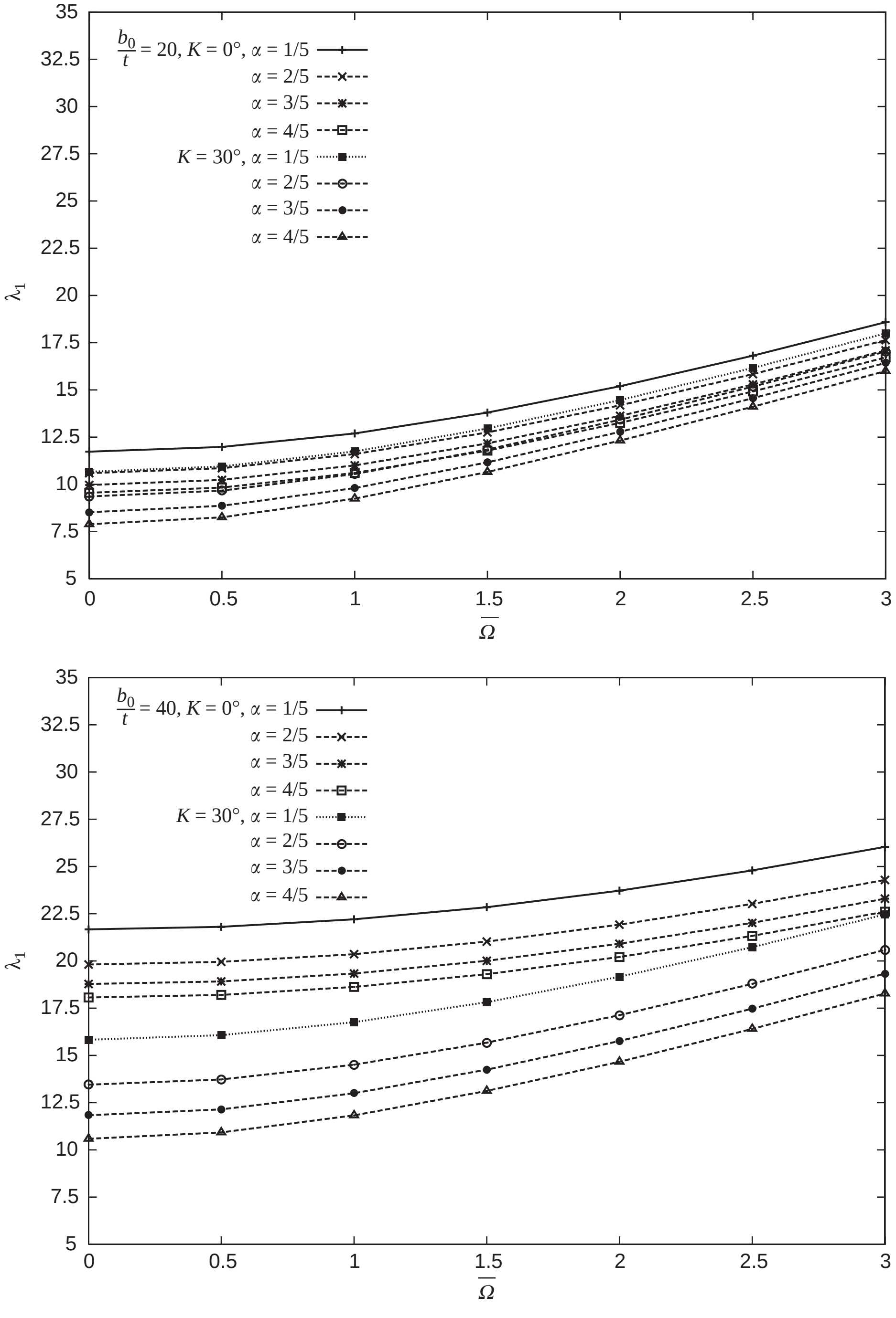

Fig. 5. Variation of λ_1 versus $\bar{\Omega}$ on various α, K and b_0/t.

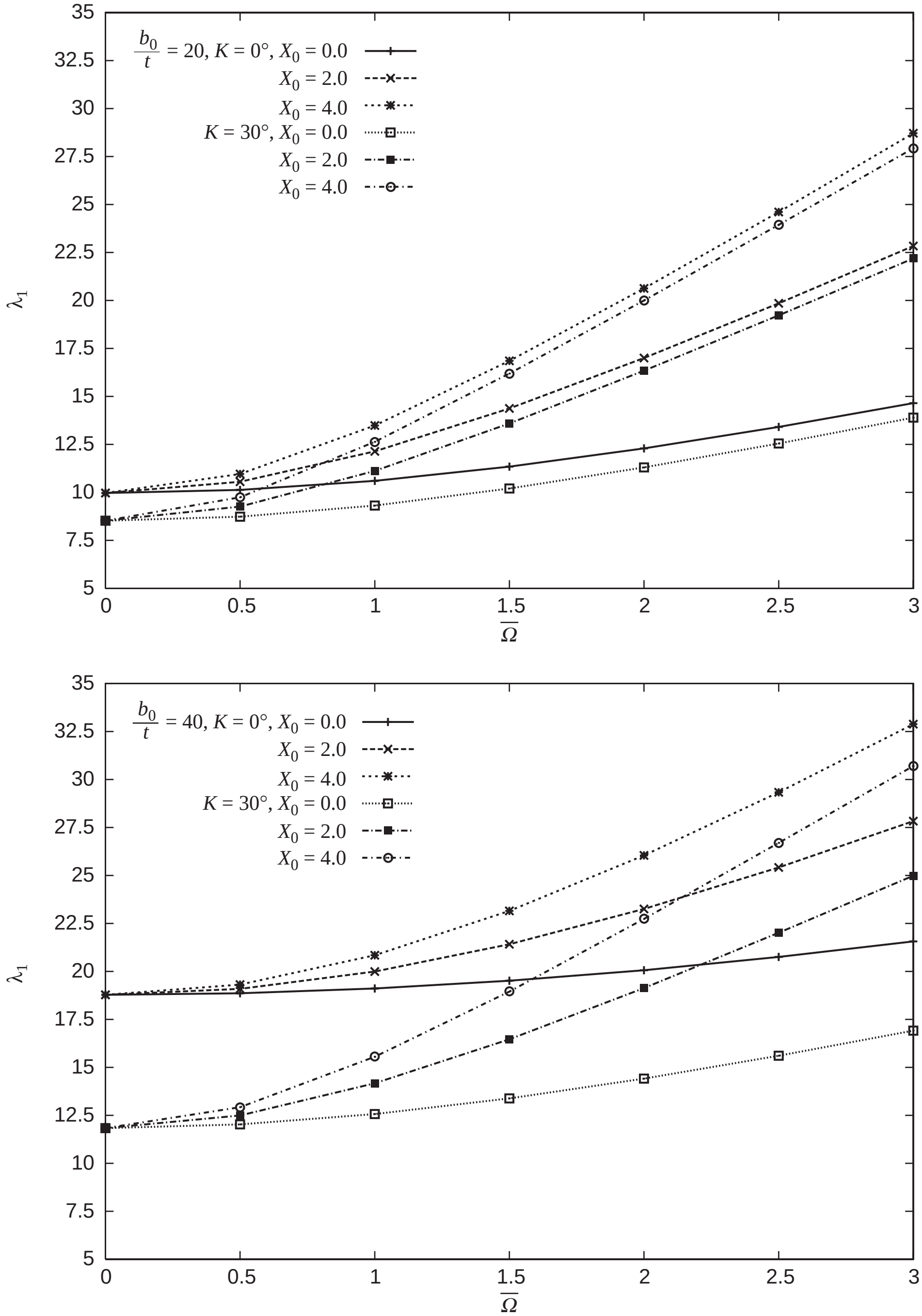

Fig. 6. Variation of λ_1 versus $\bar{\Omega}$ on various X_0, K and b_0/t.

The twist angle makes the fundamental frequency decrease for all the combinations of other parameters, but the variation of the fundamental frequency with an increase in the angular velocity becomes great due to the presence of the twist angle, which is more significant for a thin conical shell. The fundamental frequency shows an increase with the subtended angle and the radius of the hub increasing, and a decrease with an increase in the tapered ratio and the setting angle. It is observed that an increase in the angular velocity makes the fundamental frequency parameter increase and the effects of other parameters on the fundamental frequency parameter change.

Acknowledgements

An author (X. X. HU) is grateful to the Japan society for the Promotion of Science for providing a fellowship in Nagasaki University, Nagasaki, Japan.

Appendix A

The variables used in this paper are defined as the following,

$$f = a_{,x}\sin\theta - k(a\cos\theta - e), \quad h = a_{,x}\cos\theta - e_{,x} + ka\sin\theta,$$

$$p = f\sin\theta + h\cos\theta, \quad g = 1 + p^2,$$

$$q = f\cos\theta - h\sin\theta, \quad d_2 = p_{,x} + kp_{,\theta}, \quad d_3 = ap_{,x} - p_{,\theta}q,$$

$$m = -\frac{1}{a\sqrt{g}}\left(1 - \frac{d_3}{g}\right). \tag{A.1}$$

Appendix B

The non-zero elements in matrix **G** are

$$G_{1,1} = 1, \quad G_{1,2} = -\frac{q}{\bar{a}}, \quad G_{1,3} = \frac{p\bar{p}_{,x}}{g} + \frac{\bar{p}_{,x}}{g}\Phi_x^{(0)}, \quad G_{1,6} = \frac{pp_{,\theta}}{\bar{a}g} + \frac{p_{,\theta}}{\bar{a}g}\Phi_x^{(0)},$$

$$G_{1,10} = \frac{1}{\sqrt{g}}\Phi_x^{(0)},$$

$$G_{1,11} = -\frac{q}{\bar{a}\sqrt{g}}\Phi_x^{(0)}, \quad G_{1,12} = -\frac{\bar{d}_3}{\bar{a}g\sqrt{g}}, \quad G_{2,1} = \frac{1}{\bar{a}\sqrt{g}}\left(1 - \frac{\bar{a}\bar{p}_{,x}}{g}\right),$$

$$G_{2,2} = \frac{1}{\bar{a}\sqrt{g}}\left(K + \frac{\bar{p}_{,x}q}{g}\right),$$

$$G_{2,3} = \frac{1}{\bar{a}g\sqrt{g}}\left(Ka_{,x}q + p\bar{p}_{,x} + q\bar{q}_{,x} + 2\frac{p\bar{p}_{,x}\bar{d}_3}{g}\right) - \frac{\bar{p}_{,x}\bar{m}}{g}\Phi_x^{(0)},$$

$$G_{2,4} = -\frac{p_{,\theta}}{\bar{a}g\sqrt{g}}, \quad G_{2,5} = \frac{p_{,\theta}q}{\bar{a}^2g\sqrt{g}},$$

$$G_{2,6} = \frac{1}{\bar{a}^2g\sqrt{g}}\left(a_{,x}q - \bar{a}\bar{q}_{,x} + pp_{,\theta} + qq_{,\theta} + 2\frac{pp_{,\theta}\bar{d}_3}{g}\right) - \frac{p_{,\theta}\bar{m}}{\bar{a}g}\Phi_x^{(0)},$$

$$G_{2,7} = -\frac{1}{g}, \quad G_{2,8} = -\frac{q^2}{\bar{a}^2g},$$

$$G_{2,9} = \frac{2q}{\bar{a}g}, \quad G_{2,10} = \frac{p\bar{d}_3}{\bar{a}g^2} - \frac{\bar{m}}{\sqrt{g}}\Phi_x^{(0)},$$

$$G_{2,11} = -\frac{1}{\bar{a}^2g}\left(a_{,x}q - \bar{a}\bar{q}_{,x} + pp_{,\theta} + qq_{,\theta} + \frac{pq\bar{d}_3}{g}\right) + \frac{q\bar{m}}{\bar{a}\sqrt{g}}\Phi_x^{(0)},$$

$$G_{2,12} = -\frac{\bar{d}_2}{\bar{a}g^2}, \quad G_{3,4} = -\frac{p_{,\theta}}{\bar{a}^2g^2}, \quad G_{3,5} = -\frac{Kp_{,\theta}}{\bar{a}^2g^2}, \quad G_{3,7} = -\frac{1}{\bar{a}g\sqrt{g}},$$

$$G_{3,8} = \frac{Kq}{\bar{a}^2g\sqrt{g}},$$

$$G_{3,9} = \frac{1}{\bar{a}^2g\sqrt{g}}(2q - p_{,\theta}), \quad G_{3,10} = \frac{p\bar{d}_2}{\bar{a}g^2\sqrt{g}},$$

$$G_{3,11} = -\frac{1}{\bar{a}^3g\sqrt{g}}\left(a_{,x}q - \bar{a}\bar{q}_{,x} - K\bar{a}q_{,\theta} + \frac{\bar{a}pq\bar{d}_2}{g}\right),$$

$$G_{4,2} = \frac{q}{\bar{a}}, \quad G_{4,3} = \frac{a_{,x}}{\bar{a}} + \frac{K}{\sqrt{g}}\Phi_\theta^{(0)}, \quad G_{4,5} = \frac{1}{\bar{a}}, \quad G_{4,6} = -\frac{1}{\bar{a}\sqrt{g}}\Phi_\theta^{(0)},$$

$$G_{4,11} = -\frac{1}{\bar{a}}\Phi_\theta^{(0)}, \quad G_{4,12} = \frac{1}{\bar{a}\sqrt{g}},$$

$$G_{5,1} = \frac{p_{,\theta}q}{\bar{a}g\sqrt{g}}, \quad G_{5,2} = -\frac{1}{\bar{a}\sqrt{g}}\left(K + \frac{\bar{p}_{,x}q}{g}\right),$$

$$G_{5,3} = \frac{1}{\bar{a}g\sqrt{g}}(p_{,\theta}\bar{q}_{,x} - p\bar{p}_{,x} - a_{,x}\bar{p}_{,x}) - \frac{K\bar{m}}{\sqrt{g}}\Phi_\theta^{(0)},$$

$$G_{5,4} = \frac{p_{,\theta}}{\bar{a}g\sqrt{g}}, \quad G_{5,5} = \frac{1}{\bar{a}^2\sqrt{g}}\left(1 - \frac{\bar{a}\bar{p}_{,x}}{g}\right),$$

$$G_{5,6} = -\frac{2pp_{,\theta}}{\bar{a}^2 g\sqrt{g}} + \frac{\bar{m}}{\bar{a}\sqrt{g}}\Phi_\theta^{(0)}, \quad G_{5,8} = -\frac{1}{\bar{a}^2},$$

$$G_{5,10} = -\frac{p}{\bar{a}g}, \quad G_{5,11} = \frac{pq}{\bar{a}^2 g} - \frac{\bar{m}}{\bar{a}}\Phi_\theta^{(0)}, \quad G_{5,12} = -\frac{\bar{d}_2}{\bar{a}g^2},$$

$$G_{6,1} = -\frac{Kp_{,\theta}}{\bar{a}g^2}, \quad G_{6,4} = \frac{p_{,\theta}}{\bar{a}^2 g^2},$$

$$G_{6,6} = \frac{p_{,\theta}}{\bar{a}^2 g^2}\left(\frac{p\bar{d}_2}{g} - \frac{a_{,x}}{\bar{a}}\right), \quad G_{6,8} = \frac{\bar{p}_{,x}}{\bar{a}^2 g\sqrt{g}}, \quad G_{6,9} = -\frac{p_{,\theta}}{\bar{a}^2 g\sqrt{g}},$$

$$G_{6,10} = \frac{p\bar{d}_2}{\bar{a}g^2\sqrt{g}}, \quad G_{6,11} = -\frac{pq\bar{d}_2}{\bar{a}^2 g^2\sqrt{g}},$$

$$G_{7,1} = \frac{q}{\sqrt{g}}, \quad G_{7,2} = \frac{\sqrt{g}}{\bar{a}}\left(1 - \frac{q^2}{g}\right),$$

$$G_{7,3} = -\frac{1}{\bar{a}\sqrt{g}}(a_{,x}q - \bar{a}\bar{q}_{,x}) + \frac{\bar{p}_{,x}}{g}\Phi_\theta^{(0)} + \frac{K}{\sqrt{g}}\Phi_x^{(0)}, \quad G_{7,4} = \frac{1}{\sqrt{g}},$$

$$G_{7,5} = -\frac{q}{\bar{a}\sqrt{g}}, \quad G_{7,6} = -\frac{p}{\bar{a}\sqrt{g}} + \frac{\bar{p}_{,\theta}}{\bar{a}g}\Phi_\theta^{(0)} - \frac{1}{\bar{a}\sqrt{g}}\Phi_x^{(0)}, \quad G_{7,10} = \frac{1}{\sqrt{g}}\Phi_\theta^{(0)},$$

$$G_{7,11} = \frac{1}{\bar{a}}\Phi_x^{(0)} - \frac{q}{\bar{a}\sqrt{g}}\Phi_\theta^{(0)},$$

$$G_{7,12} = -\frac{2p_{,\theta}}{\bar{a}g}, \quad G_{8,1} = \frac{2q}{\bar{a}g}, \quad G_{8,2} = \frac{2}{\bar{a}g}(Kq - \bar{p}_{,x}),$$

$$G_{8,3} = \frac{4p\bar{p}_{,x}p_{,\theta}}{\bar{a}g^2} - \bar{m}\left(\frac{\bar{p}_{,x}}{g}\Phi_\theta^{(0)} + \frac{K}{\sqrt{g}}\Phi_x^{(0)}\right),$$

$$G_{8,4} = \frac{2}{\bar{a}g}, \quad G_{8,5} = -\frac{2q}{\bar{a}^2 g},$$

$$G_{8,6} = \frac{2}{\bar{a}^2 g}\left(\frac{2pp_{,\theta}^2}{g} - a_{,x}\right) - \bar{m}\left(\frac{p_{,\theta}}{\bar{a}g}\Phi_\theta^{(0)} - \frac{1}{\bar{a}\sqrt{g}}\Phi_x^{(0)}\right), \quad G_{8,8} = \frac{2q}{\bar{a}^2\sqrt{g}},$$

$$G_{8,9} = -\frac{2}{\bar{a}\sqrt{g}}, \quad G_{8,10} = \frac{2pp_{,\theta}}{\bar{a}g\sqrt{g}} - \frac{\bar{m}}{\sqrt{g}}\Phi_\theta^{(0)},$$

$$G_{8,11} = \frac{2}{\bar{a}^2\sqrt{g}}\left(a_{,x} - \frac{pp_{,\theta}q}{g}\right) - \bar{m}\left(\frac{1}{\bar{a}}\Phi_x^{(0)} - \frac{q}{\bar{a}g}\Phi_\theta^{(0)}\right),$$

$$G_{9,7} = -\frac{p_{,\theta}}{\bar{a}g^2}, \quad G_{9,8} = -\frac{1}{\bar{a}^2 g}\left(K + \frac{\bar{p}_{,x}q}{g}\right), \quad G_{9,9} = -\frac{1}{\bar{a}^2 g}\left[1 - \frac{1}{g}(\bar{a}\bar{p}_{,x} + p_{,\theta}q)\right],$$

$$G_{9,11} = \frac{1}{\bar{a}^2 g}\left[\frac{a_{,x}}{\bar{a}} + \frac{1}{g}\left(p_{,\theta}\bar{q}_{,x} - \frac{a_{,x}p_{,\theta}q}{\bar{a}} - a_{,x}\bar{p}_{,x} - Kpp_{,\theta}\right)\right], \tag{B.1}$$

where $\Phi_x^{(0)}$ and $\Phi_\theta^{(0)}$ are

$$\Phi_x^{(0)} = \frac{\bar{p}_{,x}}{g}U^{(0)} + \frac{p_{,\theta}}{\bar{a}g}V^{(0)} + \frac{1}{\sqrt{g}}\frac{\partial W^{(0)}}{\partial X} - \frac{q}{\bar{a}\sqrt{g}}\frac{\partial W^{(0)}}{\partial \theta},$$

$$\Phi_\theta^{(0)} = \frac{K}{\sqrt{g}}U^{(0)} - \frac{1}{\bar{a}\sqrt{g}}V^{(0)} + \frac{1}{\bar{a}}\frac{\partial W^{(0)}}{\partial \theta}, \tag{B.2}$$

$U^{(0)}$, $V^{(0)}$ and $W^{(0)}$ are deformation produced by the centrifugal force.

Appendix C

The variables in Eq. (13) are defined as

$$\bar{\beta}_1 = X_0 + X, \qquad \bar{\beta}_2 = \bar{a}\cos(\theta - \psi) + \bar{e}\sin\psi\cos\psi,$$

$$\bar{\beta}_3 = -\bar{a}\sin\psi\sin(\theta - \psi) - \bar{e}\sin^2\psi,$$

$$\alpha_{11} = c_{11} + fc_{21} + hc_{31}, \qquad \alpha_{12} = c_{12} + fc_{22} + hc_{32},$$

$$\alpha_{13} = c_{13} + fc_{23} + hc_{33},$$

$$\alpha_{21} = c_{21}\cos\theta - c_{31}\sin\theta, \qquad \alpha_{22} = c_{22}\cos\theta - c_{32}\sin\theta,$$

$$\alpha_{23} = c_{23}\cos\theta - c_{33}\sin\theta,$$

$$\alpha_{31} = \frac{1}{\sqrt{g}}(-pc_{11} + c_{21}\sin\theta + c_{31}\cos\theta),$$

$$\alpha_{32} = \frac{1}{\sqrt{g}}(-pc_{12} + c_{22}\sin\theta + c_{32}\cos\theta),$$

$$\alpha_{33} = \frac{1}{\sqrt{g}}(-pc_{13} + c_{23}\sin\theta + c_{33}\cos\theta),$$

$$c_{11} = 1, \quad c_{12} = 0, \quad c_{13} = -\frac{p}{\sqrt{g}}, \quad c_{21} = (f\cos\psi - h\sin\psi)\cos\psi,$$

$$c_{22} = \cos(\theta - \psi)\cos\psi,$$

$$c_{23} = \frac{1}{\sqrt{g}}\sin(\theta - \psi)\cos\psi, \qquad c_{31} = (-f\cos\psi + h\sin\psi)\sin\psi,$$

$$c_{32} = -\cos(\theta - \psi)\sin\psi,$$
$$c_{33} = -\frac{1}{\sqrt{g}}\sin(\theta - \psi)\sin\psi. \tag{C.1}$$

References

[1] J.S. Rao, Natural frequencies of turbine blades—a survey, Shock and Vibration Digest 5 (1973) 3–16.
[2] J.S. Rao, Turbine blading excitation and vibration, Shock and Vibration Digest 9 (1977) 15–22.
[3] J.S. Rao, Turbomachine blade vibration, Shock and Vibration Digest 12 (1980) 19–26.
[4] M.N. Hamdan, B.O. Al-Bedoor, Non-linear free vibrations of a rotating flexible arm, Journal of Sound and Vibration 242 (2001) 839–853.
[5] S.C. Lin, K.M. Hsiao, Vibration analysis of a rotating Timoshenko beam, Journal of Sound and Vibration 240 (2001) 303–322.
[6] S.S. Rao, R.S. Gupta, Finite element vibration analysis of rotating Timoshenko beams, Journal of Sound and Vibration 242 (2001) 103–124.
[7] G. Surace, V. Anghel, C. Mares, Coupled bending–bending–torsion vibration analysis of rotating pretwisted blades: an integral formulation and numerical examples, Journal of Sound and Vibration 206 (1997) 473–486.
[8] H.H. Yoo, Y.J. Kwak, J. Chung, Vibration analysis of rotating pre-twisted blades with a concentrated mass, Journal of Sound and Vibration 240 (2001) 891–908.
[9] V. Ramamurti, R. Kielb, Natural frequencies of twisted rotating plates, Journal of Sound and Vibration 97 (1984) 429–449.
[10] S. Sreenivasamurthy, V. Ramamurti, A parametric study of vibration of rotating pre-twisted and tapered low aspect ratio cantilever plates, Journal of Sound and Vibration 76 (1981) 311–328.
[11] A.W. Leissa, J.K. Lee, A.J. Wang, Rotating blade vibration analysis using shells, Journal of Engineering for Power, Transactions of American Society of Mechanical Engineers 104 (1982) 296–302.
[12] T. Tsuiji, M. Shugyo, T. Yamashita, Free vibration analysis of rotating thin twisted plates, Transactions of Japan Society of Mechanical Engineers 61 (C) (1995) 4575–4586 (in Japanese).
[13] X.X. Hu, T. Tsuiji, Free vibration analysis of rotating twisted cylindrical thin panels, Journal of Sound and Vibration 222 (1999) 209–224.